1. $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}(x-a) = 0$

where a, b, g are being positive numbers, and x=a', $\frac{dx}{dt} = 0$ when t=0, then prove that

$x = a + (a'-a)cos\sqrt{\dfrac{g}{b}}t$

The given equation can be written as

$(D^2 + \dfrac{g}{b})x = \dfrac{ga}{b}$

Auxiliary Equation $m^2 + (\dfrac{g}{b}) = 0 \qquad => m = \pm\sqrt{\dfrac{g}{b}}$

$C.F. = c_1 cos\sqrt{\dfrac{g}{b}}t + c_2 sin\sqrt{\dfrac{g}{b}}t$

$$P.I = \frac{1}{D^2 + \frac{g}{b}} * \frac{ga}{b}$$
$$= \frac{1}{D^2 + \frac{g}{b}} * \frac{ga}{b}e^{0t}$$
$$= \frac{ga}{b}.\frac{e^{0t}}{0 + g/b} = \frac{ga}{b}\frac{b}{g} = a$$

Complete Solution $c_1 cos\sqrt{\dfrac{g}{b}}t + c_2 sin\sqrt{\dfrac{g}{b}}t + a$

Now, $\dfrac{dx}{dt} = -c_1\sqrt{\frac{g}{b}}sin(\sqrt{\frac{g}{b}}t) + c_2\sqrt{\frac{g}{b}}cos(\sqrt{\frac{g}{b}}t)$

As $x = a' \quad when \;\; t = 0$ therefore, from complete solution,

$a' = c_1 + a$ or $c_1 = a' - a$

$\dfrac{dx}{dt} = 0$ when $t = 0$, so we have

$0 = c_2\sqrt{\frac{g}{b}}$ or $c_2 = 0$

Putting value of c_1 and c_2 we get

$x = a + (a'-a)cos\sqrt{\frac{g}{b}}t$

2. Find the general solution of the equation $y" + 4y' + 3y = xsin2x$

The characteristic equation of the corresponding homogenous equation is

$m^2 + 4m + 3 = 0, \quad or \quad (m+1)(m+3) = 0$ Its roots are $m = -1, \;\; -3$

The complementary function is $y_c = Ae^{-x} + Be^{3x}$

The particular integral is

$y_p(x) = \dfrac{1}{[D^2 + 4D + 3]}(xsin2x)$= Imaginary part of$\dfrac{1}{[D^2 + 4D + 3]}(e^{2ix}x)$

$$= Ime^{2ix}\frac{1}{[(D+2i)^2+4(D+2i)+3]}x$$
$$= Ime^{2ix}[D^2+4(1+i)D+(8i-1)]^{-1}(x)$$
$$= Im\frac{e^{2ix}}{8i-1}[1+\frac{4(1+i)D}{8i-1}+\frac{D^2}{8i-1}]^{-1}x$$
$$= Im\frac{e^{2ix}}{8i-1}[1-\frac{4(1+i)D}{8i-1}+...](x)$$
$$= Im\frac{(8i+1)}{(-65)}e^{2ix}[x-\frac{4(8i+1)(1+i)}{-65}]$$
$$= Im[-\frac{1}{65}(8i+1)(cos2x+isin2x)[x+\frac{4}{65}(9i-7)]]$$
$$= Im[-\frac{1}{65}[(cos2x-8sin2x)+i(sin2x+8cos2x)][(x-\frac{28}{65})+\frac{36}{65}i]]$$
$$= -\frac{1}{4225}[65x(8cos2x+sin2x)-28(8cos2x+sin2x)+36(cos2x-8sin2x)]$$
$$= -\frac{1}{4225}65x(8cox2x+sin2x)-188cos2x-316sin2x]$$

The general solution is

$$y(x) = Ae^{-x}+Be^{-3x}-\frac{1}{4225}65x(8cox2x+sin2x)-188cos2x-316sin2x]$$

3. Find the general solution of the equation $y^{iv}+3y" = 108x^2$

The characteristic equation of the homogeneous equation is

$m^4+3m^2=0,$ or $m^2(m^2+3)=0.$ Its roots are $m=0,\ 0\pm\sqrt{3}i$

The complimentary function is $y_c(x) = A+Bx+(Ccos\sqrt{3}x+Dsin\sqrt{3}x)$

We have $F(D) = D^4+3D^2 = D^2(D^2+3)$. The particular integral is given by

$$y_p(x) = 108[D^2(D^2+3)^{-1}(x^2)] = 108[D^{-2}]\frac{1}{3}[1+\frac{D^2}{3}]^{-1}(x^2)$$
$$= 36[D^{-2}][1-\frac{D^2}{3}+\frac{D^4}{9}-...](x^2) = 36D^{-2}[x^2-\frac{2}{3}]$$
$$= 36[\frac{x^4}{12}-\frac{x^2}{3}] = 3x^4-12x^2.$$

The general solution is $y(x) = A+Bx+(Ccos\sqrt{3}x+Dsin\sqrt{3}x)+3x^4-12x^2$

4. Find the general solution of the equation $y"-4y'+13y = 18e^{2x}sin3x$

The characteristic equation of the homogeneous equation is

$m^2-4m+13=0.$ Its roots are $m = \frac{4\pm\sqrt{16-52}}{2} = 2\pm3i$

The complimentary function is $y_c(x) = e^{2x}(Acos3x + Bsin3x)$.

We have $F(D) = D^2 - 4D + 13$ and $r(x) = 18e^{2x}sin3x$.

We write particular integral as

$$
\begin{aligned}
y_p(x) &= 18\frac{1}{(D^2-4D+13)^{-1}}(e^{2x}sin3x) \\
&= 18e^{2x}\frac{1}{(D+2)^2-4(D+2)} + 13(sin3x) \\
&= 18e^{2x}\frac{1}{(D^2+9)}sin3x \quad = \quad 18e^{2x}Im\frac{1}{D^2+9}(e^{3ix}) \\
&= 18e^{2x}Im\ e^{3ix}\frac{1}{(D+3i)^2+9}(1) \\
&= 18e^{2x}Im\ e^{3ix}[D^2+6iD]^{-1}(1) \\
&= 18e^{2x}Im\ e^{3ix}D^{-1}(D+6i)^{-1}(1) \\
&= 3xe^{2x}Ime^{3ix}D^{-1}(0+61)^{-1}(1) \quad = \quad 18e^{2x}Im\frac{1}{6i}xe^{3ix} \\
&= 3xe^{2x}Im{-i}(cos3x + isin3x) \quad = \quad -3xe^{2x}cos3x
\end{aligned}
$$

The general solution is $y(x) = e^{2x}(Acos3x + Bsin3x) - 3xe^{2x}cos3x$

5. Find the general solution of the equation $x^2y" + 5xy' + 3y = lnx \quad, x > 0$. Using the transformation $x = e^t$, we obtain

$$
\frac{d^2y}{dt^2} - \frac{dy}{dt} + 5\frac{dy}{dt} + 3y = ln(e^t), \ \ or \ \ \frac{d^2y}{dt^2} + d\frac{dy}{dt} + 3y = t. \tag{1}
$$

The characteristic equation of the corresponding homogeneous equation is

$m^2 + 4m + 3 = 0$, or $(m+1)(m+3) = 0$, or $m = -1, -3$

The complimentary function is $y_c(t) = Ae^{-t} + Be^{-3t}$.

Let the particular integral be written as $y_p = c_1t + c_2$. Substituting in Eqn(1), we get

$$4c_1 + 3(c_1t + c_2) = t$$

Comparing the coefficients of t and the constant terms on both sides, we obtain $3c_1 = 1$ and $4c_1 + 3c_2 = 0$. The solution is $c_1 = 1/3$, $c_2 = -4/9$

The particular integral is $y_p = \frac{t}{3} - \frac{4}{9}$. The general solution is

$$y(t) = Ae^{-t} + Be^{-3t} + \frac{t}{3} - \frac{4}{9}$$

Substituting $e^t = x$, we get the general solution as

$$y(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{1}{3}lnx - \frac{4}{9}$$

6.